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Some Results on the Kernel of a Set

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1. INTRODUCTION

A set S in a linear space is said to be *star-shaped* relative to a point a if, for every $x \in S$, the line segment $[a, x] = \{\lambda a + (1 - \lambda) x: 0 \leqslant \lambda \leqslant 1\}$ is a subset of S. The set of all points relative to which S is star-shaped is called the *kernel* of S and is denoted S^*. Brunn [2] showed that the kernel S^* is a convex set; see, e.g., Valentine [6, Theorem 1.2]. We accept the possibility that S^* is empty. Recently, Bryant [3] generalized Brunn's result to associative geometries and at the same time provided a simple proof.

The purpose of this paper is to prove that the kernel of a set is precisely the intersection of all maximal convex subsets of S. Such a result has been proved for linear spaces by Toranzos [5]. We do this for spaces which are more general than linear spaces but more restrictive than associative geometries. We also show that the family which produces the kernel through an intersection can be made smaller than the family of all maximal convex subsets. This last result is then applied to linear topological spaces.

2. A PRELIMINARY RESULT

We use the notation of Bryant [3]. Let X be a nonempty set. With each ordered pair (a, b) of elements of X, associate a subset $a \cdot b$ (or simply ab) of X. For $A \subset X$ and $B \subset X$, define

$$AB = \bigcup \{ab: a \in A, b \in B\}.$$

We will abbreviate $\{a\}B$ as aB and $A\{b\}$ as Ab. We say that $(X, \cdot)$ is an *associative geometry* if $a(bc) = (ab) c$ for all a, b and c in X. If X is an associative geometry and A, B, C, D are subsets of X, then it is easy to see that

$$(AB) C = A(BC), \qquad A(B \cup C) = (AB) \cup (AC),$$

and that $A \subset C$, $B \subset D \Rightarrow AB \subset CD$.

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Let X be an associative geometry and let $S \subset X$. Then S is said to be *star-shaped* relative to $a \in S$ if $aS \subset S$. Further, S is said to be *convex* if $SS \subset S$. Note that these definitions reduce to the usual ones for linear spaces if one takes ab to be the line segment $[a, b]$. The *kernel* S^* of S is defined to be the set of all $a \in S$ relative to which S is star-shaped. In symbols, $S^* = \{a \in S: aS \subset S\}$. Bryant [3] and Bryant and Webster [4] have shown that S^* is convex for each S.

For our results in the next section we impose additional restrictions on the geometry. An associative geometry X is called *Abelian* if $ab = ba$ for all a, b in X. A part of the following theorem corresponds to Theorem 1.25 in [6]; see also Theorem 5 in [4].

THEOREM 1. *Let A and B be convex subsets of an Abelian associative geometry X. Then AB and $AB \cup A \cup B$ are both convex.*

Proof. It is easy to see that the operation $\cdot$ is commutative even when acting on subsets of X. Therefore,

$$(AB)(AB) = (AA)(BB) \subset AB,$$

which shows that AB is convex. Now let $C = AB \cup A \cup B$. Then

$$(AB)\, C = [(AB)(AB)] \cup [(AB)\, A] \cup [(AB)\, B] \subset AB.$$

Further,

$$AC = [A(AB)] \cup [AA] \cup [AB] \subset [AB \cup A].$$

Similarly

$$BC \subset [(AB) \cup B].$$

Putting the three steps together, we see that

$$CC \subset [AB \cup A \cup B] = C.$$

This shows that C is convex and completes the proof of the theorem.

The following example shows that the above theorem may fail if the Abelian property does not hold.

EXAMPLE 1. Let $X = \{1, 2, 3\}$ and define $xy = \{x, y\}$ if $x \leqslant y$ and $xy = \{x, y, 1\}$ if $x > y$. It can be checked that $(xy)\, z = x(yz) = \{x, y, z\}$ if $x \leqslant y \leqslant z$ and that $(xy)\, z = x(yz) = \{x, y, z, 1\}$ in all other cases. Thus we have an associative geometry. Now $A = \{2\}$ and $B = \{3\}$ are both convex but $AB = \{2, 3\} = (AB) \cup A \cup B$ is not convex, because $3.2 = X$ is not a subset of AB. Thus, Theorem 1 fails.

3. The Relation Between the Kernel and Maximal Convex Subsets

Throughout this section, S will denote a set in an associative geometry X. A subset F of S is called a ***maximal convex subset*** if F is convex and if F is not a proper subset of any convex subset of S. The symbol $\mathscr{F}_S$ will denote the family of all maximal convex subsets of S. As before, S^* will denote the kernel of S. To establish the desired connection between S^* and $\mathscr{F}_S$ we need the following (somewhat natural) condition.

Condition (a). *Every singleton subset of X is convex.* This condition means that $xx = \{x\}$ or $\varnothing$ for all $x \in X$. Note that if X is a linear space and xy is defined as the line segment $[x, y]$, then Condition (a) is obviously satisfied.

It is a simple consequence of the maximality principle that, given any convex subset A of S, there is always a maximal convex subset F of S such that $F \supset A$. This fact is used in Theorem 2.

Theorem 2. *Let X, S, S^* and $\mathscr{F}_S$ be as stated above. Let*

$$T_S = \bigcap \{F : F \in \mathscr{F}_S\}.$$

(i) *If X is Abelian, then $T_S \supset S^*$;*

(ii) *If X satisfies Condition* (a), *then $T_S \subset S^*$.*

(iii) *If X is Abelian and satisfies Condition* (a), *then $T_S = S^*$.*

Proof. (1) Suppose X is Abelian. Let $F \in \mathscr{F}_S$. Then F is convex and S^* is also convex. Therefore, by Theorem 1, $G = (FS^*) \cup F \cup S^*$ is convex. But FS^*, F, and S^* are all subsets of S. Therefore, G is a convex subset of S such that $G \supset F$. The maximality of F shows that $F = G$. But then $F \supset S^*$. It follows that $T_S \supset S^*$. Thus, (i) is proved.

(2) Suppose that condition (a) holds. Let $x \in S \backslash S^*$. Then there is a $y \in S$ such that $xy \not\subset S$. Now $\{y\}$ is convex and hence there is an $F \in \mathscr{F}_S$ such that $y \in F$. But then $x \notin F$, for otherwise $xy \subset F \subset S$, which is a contradiction. Thus, $x \notin T_S$. Thus, $T_S \subset S^*$. This proves (ii) and completes the proof of the theorem.

As noted in Section 1, the special case of Theorem 2(iii) applicable to linear spaces has been proved by Toranzos [5]. We give examples below to show that the theorem may fail if either the Abelian property or condition (a) is dropped.

Example 2. In the set-up of Example 1, $X = \{1, 2, 3\}$, $xy = \{x, y\}$ if $x \leqslant y$ and $xy = \{x, y, 1\}$ if $x > y$. Let $S = \{2, 3\}$. Then, as seen earlier S

is not convex. Therefore $\{2\}$ and $\{3\}$ are the only maximal convex subsets of S. It follows that $T_S = \varnothing$. But $S^* = \{2\}$. Thus, Theorem 2(i) fails. Condition (a) is satisfied but the Abelian property fails.

EXAMPLE 3. Let $X = \{1, 2, 3, 4\}$. Define an Abelian operation as follows: $xy = \{x, y\}$ if $x \leqslant 2$ and $y \leqslant 2$; $1 \cdot 3 = 3 \cdot 1 = \{1, 2, 3\}$ and $xy = X$ for all other pairs (x, y). It is tedious but straightforward to verify that the operation is associative. Let $S = \{1, 2, 3\}$. Then $S^* = \{1\}$. But the only maximal convex subset of S is $\{1, 2\}$. Therefore, $T_S = \{1, 2\} \neq S^*$. Thus, Theorem 2(ii) fails. The Abelian property holds but Condition (a) fails.

Again let S be a set in an associative geometry X. For $y \in S$, the set $B(y, S) = \{x \in S: xy \not\subset S\}$ may be called the *blind subset* of y relative to S. It is clear that $B(y, S) \subset S \backslash S^*$. It is possible that $B(y, S)$ is empty.

THEOREM 3. *Let X, S, S^*, $\mathscr{F}_S$ and T_S be as in Theorem 2. Let $G \subset S$ be such that the family $\{B(y, S): y \in G\}$ covers $S \backslash S^*$. Let a subfamily $\mathscr{F}$ of $\mathscr{F}_S$ cover G and let $V = \bigcap \{F: F \in \mathscr{F}\}$. Then $V \subset S^*$. Moreover, if X is Abelian, then $V = S^*$.*

Proof. Let $x \in S \backslash S^*$. Then there is a $y \in G$ such that $x \in B(y, S)$. That is, $xy \not\subset S$. Since $y \in G$, there is an $F \in \mathscr{F}$ such that $y \in F$. But then $x \notin F$ and hence $x \notin V$. Thus, $V \subset S^*$, and the first assertion is proved. Since $V \supset T_S$ trivially, the second assertion follows from the first assertion and Theorem 2(i). The proof is complete.

Suppose that the Abelian property holds. Then it is easy to see that $\bigcup \{B(y, S): y \in S \backslash S^*\}$ equals $S \backslash S^*$. Thus, in applying Theorem 3, we may take $G = S \backslash S^*$. But Theorem 2(i) shows that a subfamily $\mathscr{F}$ of $\mathscr{F}_S$ covers $S \backslash S^*$ if, and only if, it covers the whole of S. We thus get the following corollary.

COROLLARY 1. *Let X, S, S^* and $\mathscr{F}_S$ be as in Theorem 3. If X is Abelian and a subfamily $\mathscr{F}$ of $\mathscr{F}_S$ covers S, then $\bigcap \{F: F \in \mathscr{F}\} = S^*$.*

Corollary 1 can be considered to be a generalization of Theorem 2(iii), because Condition (a) implies that $\mathscr{F}_S$ covers S. For linear spaces the result of Corollary 1 has been given by Bragard [1].

COROLLARY 2. *Let S be a closed subset of a linear topological space. If a subfamily $\mathscr{F}$ of $\mathscr{F}_S$ covers the boundary of S, then $\bigcap \{F: F \in \mathscr{F}\} = S^*$.*

Proof. Let G denote the boundary of S. Let $x \in S \backslash S^*$. Then there is a $z \in S$ such that the line segment $[x, z] \not\subset S$. Let $y(\lambda)$ denote the point $(1 - \lambda) x + \lambda z$. If $D = \{\lambda: 0 < \lambda < 1 \text{ and } y(\lambda) \notin S\}$, then D is open and non-empty. Let $\lambda^* = \sup D$ and $y^* = y(\lambda^*)$. Then it is clear that $y^* \in G$ and that $[x, y^*] \not\subset S$. Thus $x \in B(y^*, S)$. In other words, $\{B(y, S): y \in G\}$ covers $S \backslash S^*$. The corollary now follows from Theorem 3.

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